

# COURANT ALGEBROIDS

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ABSTRACT. This paper is devoted to studying some properties of the Courant algebroids: we explain the so-called "conducting bundle construction" and use it to attach the Courant algebroid to Dixmier-Douady gerbe (following ideas of P. Severa). We remark that WZNW-Poisson condition of Klimcik and Strobl (math.SG/0104189) is the same as Dirac structure in some particular Courant algebroid. We propose the construction of the Lie algebroid on the loop space starting from the Lie algebroid on the manifold and conjecture that this construction applied to the Dirac structure above should give the Lie algebroid of symmetries in the WZNW-Poisson  $\sigma$ -model, we show that it is indeed true in the particular case of Poisson  $\sigma$ -model.

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## 1. INTRODUCTION

In this note we study some properties of Courant algebroids and argue their natural connection to gerbes, WZNW-Poisson manifolds and possibly Poisson  $\sigma$ -model.

Courant algebroids introduced in [11] are certain kinds of algebroids for which the conditions antisymmetry, Leibniz rule and the Jacobi identity for the bracket are relaxed in a certain way. Original motivation of [11] was to characterize Poisson and symplectic structure in the same manner, such that the graph of the map given by the Poisson structure from the 1-forms (from the vector fields in the symplectic case) to the direct sum of 1-forms and vector fields is subalgebra with respect to the Courant bracket. Further investigation [18] showed that the Courant algebroid is very natural setup for the construction of bialgebroids. Also Courant algebroids are relevant to studying dynamical  $r$ -matrices [19] and have various other applications (see [24] and references therein).

Let us recall the Courant bracket on  $T^*M \oplus TM$ :

$$[\alpha_1 + \xi_1, \alpha_2 + \xi_2] = (\mathcal{L}_{\xi_1}\alpha_2 - \mathcal{L}_{\xi_2}\alpha_1 + \frac{1}{2}d(\iota_{\xi_2}\alpha_1 - \iota_{\xi_1}\alpha_2)) + [\xi_1, \xi_2] \quad (1.0.1)$$

The bracket is antisymmetric, but the Leibniz rule and Jacobi identity are not true, but the anomalies for them are easily expressible in terms of the pairing:

$$\langle \alpha_1 + \xi_1, \alpha_2 + \xi_2 \rangle = (\iota_{\xi_2} \alpha_1 + \iota_{\xi_1} \alpha_2) \quad (1.0.2)$$

In our text (except for the section 4.5) we will use slightly modified bracket:

$$\begin{aligned} [\alpha_1 + \xi_1, \alpha_2 + \xi_2] &= (\mathcal{L}_{\xi_1} \alpha_2 - \mathcal{L}_{\xi_2} \alpha_1 + d(\iota_{\xi_2} \alpha_1)) + [\xi_1, \xi_2] = \\ &= \mathcal{L}_{\xi_1} \alpha_2 - \iota_{\xi_2} d\alpha_1 + [\xi_1, \xi_2] \end{aligned} \quad (1.0.3)$$

This bracket is not antisymmetric but one has the left Jacobi identity and right Leibniz rule for it. This was discovered by Y. Kosmann-Schwarzbach, P. Severa and one of the authors (P.B.) independently (unpublished). The brackets are connected to each other by adding the  $\frac{1}{2} \langle, \rangle$ .

*Definition:* let us call the bundle  $T^*M \oplus TM$  with the bracket 1.0.3 and anchor map  $\pi : \alpha + \xi \mapsto \xi$  the *standard Courant algebroid*.

We refer to [18] or to the section 2.5 of the present paper for the general definition of Courant algebroids.

The paper organized as follows in section 2 we recall some basic definitions.

In section 3 we realize in details the "conducting bundle" construction (idea of which is due to P. Severa). The construction is the following: to any extension

$$0 \rightarrow \mathcal{O} \rightarrow \tilde{\mathcal{T}} \xrightarrow{\sigma} \mathcal{T} \rightarrow 0 \quad (1.0.4)$$

we associate the Courant algebroid in a canonical and functorial way. The main property of this construction is the following (see section 3.3.2): assume there was an algebroid structure on the extension above then one has canonical trivialization of the obtained Courant algebroid. In smooth situation (opposite to holomorphic) all extension can be endowed with the bracket, hence all Courant algebroids obtained this way are the standard ones. But even in smooth situation the construction is not useless. Following P. Severa, this construction can be used to attach a Courant algebroid to a gerbe. Moreover morally speaking the relation of gerbes to Courant algebroids is the same as of line bundles to their Atiyah algebroids. But the precise formulation of this analogy is yet unknown.

In section 4 we study the properties of the exact Courant algebroids, this is subclass of Courant algebroids which are the extensions of the type:

$$0 \rightarrow \Omega^1 \rightarrow \mathcal{Q} \xrightarrow{\pi_{\mathcal{Q}}} \mathcal{T} \rightarrow 0 \quad (1.0.5)$$

This kinds of algebroids can be obtained by the construction of the previous section. P. Severa noticed the following: deformations of such algebroids are governed by the closed 3-forms. This is quite striking fact. He also proposed to mimic the differential geometry for the Courant algebroids i.e. to introduce the connections, their curvature and prove the basic properties analogous to the classical situation e.g. the connections are the affine space over the 2-forms, the curvature is closed 3-form and etc. In particular using these properties one can see that there is natural structure of  $\mathcal{O}$ -module on the Courant algebroid obtained by the previous construction.

At the end of this section we make a remark that WZNW-Poisson condition of Klimcik and Strobl [16] coincides with Dirac structure in the Courant algebroid twisted by the 3-form (this was discovered independently by P. Severa and A. Weinstein [27]).

Section 5 is motivated by the papers [16, 17] and we try to give more geometric understanding of some of their constructions. Consider the diagram:

$$\begin{array}{ccc} \text{Map}(X, Y) \times X & \xrightarrow{ev} & Y \\ \text{pr} \downarrow & & \\ \text{Map}(X, Y) & & \end{array} \quad (1.0.6)$$

We will show that if  $A$  is an algebroid on  $Y$  then there  $pr_*ev^*A$  has a natural structure of algebroid, considering the  $T^*M$  as an algebroid with the bracket constructed from Poisson bracket on  $M$  and considering its pullback  $pr_*ev^*T^*M$  on the loop space  $LM$  we see that this algebroid is the same as algebroid considered in [17] which plays the role of symmetries in Poisson  $\sigma$ -model. Analogous algebroids were considered in [9, 10]. We also hope that for a Courant algebroid on  $M$  one can naturally construct Lie algebroid on  $LM$  this construction should be agreed with the construction of line bundle on  $LM$  from gerbe on  $M$  and the transgression of characteristic class of gerbe to characteristic class of line bundle on  $LM$ .

**Remark on Notations.**

In this work will denote by  $\mathcal{O}$  functions on our manifolds. Most of the considerations works for analytic or algebraic as well as for smooth functions. The basic situation in our paper that  $\mathcal{O}$  are smooth functions. We are sorry, that it may cause some inconvenience for those readers, who get used to think about  $\mathcal{O}$  only as about analytic or algebraic functions. The same words should be said about our notation  $\mathcal{T}$  - it is tangent bundle, basic example is smooth tangent bundle for smooth manifold, but most considerations works well for the holomorphic tangent bundle as well.

**Remark on Further Developments.**

This paper was basically finished in summer 2001. Since that time there have been some further developments in the subject. One of the authors (P.B.) in [28] proposed the analogy between Courant algebroids and “vertex algebroids” i.e. algebroids that appeared in [22, 23] and which gives rise to the chiral de Rham complex and also he gave a “coordinate free” construction and proved the uniqueness of the vertex algebroid. The paper [27] contains many important ideas related to the subject of present work.

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## 2. PRELIMINARY DEFINITIONS

For the convenience of the reader let us recall definitions of Lie-algebroids, Atiyah algebroid, gerbes, Courant algebroids.

**2.1. Lie algebroids.** *Definition:* Let  $M$  be a smooth (complex) manifold. A *Lie algebroid* is a vector bundle  $A$  endowed with a Lie algebra bracket  $[\cdot, \cdot]_A$  on the space of sections  $\Gamma(A)$  and an anchor map  $p : A \rightarrow TM$ , which induces a Lie algebra homomorphism  $\Gamma(A) \rightarrow Vect(M)$  satisfying the Leibniz rule

$$[X, fY]_A = f[X, Y]_A + (a(X)f)Y$$

We refer to [21] for the comprehensive study of Lie algebroids and groupoids, see also [6]. In a sense Lie algebroid is a generalization of the notion of tangent bundle, which is the example of Lie algebroid which should be kept in mind first of all. The natural examples of Lie algebroids are Atiyah algebroid and the algebroid structure on  $T^*M$  constructed from Poisson bracket on  $M$ . The notion of de-Rham complex, Lie derivatives and etc. holds identically true for any Lie algebroid. One is referred to [13] for the interesting studies of this theory.

**2.2. Atiyah algebroid.** Atiyah algebroid is the natural example of Lie-algebroid constructed from any bundle on  $M$  (or speaking algebraically from projective module over some commutative algebra  $A$ .) We will use this construction in order to explain the relation of Courant algebroid to gerbes. Informally speaking, Courant algebroid should be an analogue or somehow related to Atiyah algebroid, not for the bundle, but for the gerbe, but the precise formulation is not yet known.

The idea of construction of Atiyah algebroid goes back to the paper [2]. It was studied for example in [3, 26]. For very nice modern exposition of Atiyah classes see first paragraphs of [15]. The construction associates to any vector bundle a Lie algebroid, which morally plays the role of the Lie algebra of group of local automorphisms of the bundle. The construction is functorial with respect to automorphisms of vector bundle. The Atiyah algebroid is the space of derivations of module of sections of bundle. Let us give formal definitions below.

*Definition:* let us call the *derivation of the module*  $P$  over some commutative ring  $A$  the pair of maps  $d : A \rightarrow A$  and  $\bar{d} : P \rightarrow P$  such that  $d$  is derivation of algebra  $A$  and  $\bar{d}$  is derivation of  $P$ , which lifts  $d$  i.e.  $\bar{d}(ap) = a\bar{d}p + d(a)p$ , for  $a \in A$  and  $p \in P$ .

Let us consider some manifold  $M$  and the vector bundle  $F$  over it. Let us denote the algebra of functions on  $M$  by  $A$  and the module of sections of  $F$  by  $P$ .

*Definition:* *Atiyah algebroid*  $A(F)$  (also denoted  $Der(F)$ ) of the bundle  $F$  is the set of all derivations of the module  $P$ .

(In algebraic or holomorphic situation we should add the word sheaf everywhere).

*Lemma:* Obviously the following is true:

1.  $A(F)$  is naturally  $A$ -module.
2.  $A(F)$  is naturally Lie algebra, with respect to commutator of maps.
3. the "anchor" map  $\bar{d} \rightarrow d$  is Lie algebra homomorphism
4. There is the following exact sequence of Lie algebras.

$$End(P) \rightarrow A(F) \rightarrow \mathcal{T}_M \tag{2.2.1}$$

where  $End(P)$  is the space  $A$ -linear endomorphisms of  $P$ , and  $\mathcal{T}_M$  is the module of vector fields (derivations) on  $M$ .

*Warning:* note the derivations of  $A$  considered as an algebra and as a module over itself are not the same, there is an exact sequence

$$A \rightarrow Der_{module} A \rightarrow Der_{algebra} A$$

*Remark:* the *connection* on the bundle  $F$  has a nice interpretation as a splitting  $\nabla : \mathcal{T}_M \rightarrow A(F)$  of the sequence 2.2.1. We mean the splitting is homomorphism of  $A$ -modules. The curvature is obviously the measure how far is the splitting from the homomorphism of Lie algebras. The Bianchi identity follows from Jacobi identity. (See [1] for the nice expositions of this matters).

*Lemma:* the association  $F \rightarrow A(F)$  is functorial with respect to automorphisms of  $F$ .

Actually, if  $\Phi$  is an automorphism of  $A$ -module  $M$  and  $d$  is a derivation of  $M$ , then  $\Phi d \Phi^{-1}$  is again derivation of  $M$ .

*Example:* let us consider the linear bundle  $L$  over  $M$ , then its Atiyah algebroid is algebroid of the type:  $\mathcal{O} \xrightarrow{i} A(L) \xrightarrow{p} \mathcal{T}_M$

One knows that such algebroids are classified by the  $H^2(M)$ .

*Example:* If an invertible function  $g$  gives an automorphism of  $L$ , then induced automorphism of  $A(L)$  is given by the 1-form  $g^{-1}dg = d\log(g)$ .

*Lemma:* The association of  $L \rightarrow A(L)$  on the level of cocycles corresponds to the map  $f \mapsto d\log(f)$ , where  $f \in H^1(\mathcal{O}^*)$  (i.e. in Čech description it is invertible function on the intersection of charts) and  $d\log(f)$  is closed 1-form on the intersection of charts i.e. it corresponds to Čech-deRham description of  $H^2(M)$ .

*Remark:* Let us make a remark that the extension  $\mathcal{O} \rightarrow A(L) \rightarrow \mathcal{T}$  is not trivial extension of coherent sheaves in general (see [2]) (we mean holomorphic situation, in smooth case all extensions are trivial). The triviality of extensions means that there exists holomorphic connection (because as it was explained above the splitting of this extension is the same as connection on  $L$ .)

**2.3. Lie Algebroid on  $T^*M$  from Poisson bracket on  $M$ .** Let  $M$  be a manifold and  $\pi$  be Poisson bracket on it. Then there is natural structure of the Lie algebroid on the  $T^*M$ . We will use it in relation to WZNW-Poisson  $\sigma$ -model.

The Poisson bracket on functions can be naturally extended to the bracket on 1-forms [12] defined by the rules  $[df, dg] = d\{f, g\}$ , and Leibniz rule for functions  $[hdf, dg] = \{h, g\}df + h[df, dg]$ ;  $[f, dg] = \{f, g\}$ . The bracket can be written explicitly as follows:  $[\alpha, \beta] = \mathcal{L}_{\langle \pi, \alpha \rangle} \beta - \mathcal{L}_{\langle \pi, \beta \rangle} \alpha - d\langle \pi, \alpha \wedge \beta \rangle$ .

This bracket supplied with the anchor map  $p : T^*M \rightarrow TM$ , by the rule  $\alpha \mapsto \langle \pi, \alpha \rangle$  gives the  $T^*M$  the structure of Lie algebroid.

Where we denoted by  $\langle \pi, \alpha \rangle$  a vector which is the contraction of bivector  $\pi$  and 1-form  $\alpha$ . (One can check that the anchor is really a homomorphism of Lie algebras).

**2.4. Gerbes.** Gerbes were introduced by Giraud [7] in early seventies. Recently they were found to be useful in different problems of mathematics and physics (see [4, 5, 14, 8]). Informally speaking gerbes (more precisely Dixmier-Douady gerbes) are higher analogues of the linear bundles, i.e. line bundles are classified by the  $H^2(X, \mathbb{Z})$  and gerbes are the objects classified by  $H^3(X, \mathbb{Z})$ . There two ways to treat gerbes one is more concrete and is similar

to defining the line bundles by the gluing functions on each chart (see [14]). Another is more abstract - the language of sheaves of categories (see [4]). We refer the reader to section 1 of [5] for short and concise discussion of the material and for the comparison of the two approaches.

*Definition:* a sheaf of groupoids is called *Dixmier-Douady gerbe* (or DD-gerbe) if it satisfies the following:

- 1) all objects  $\mathcal{F}(U)$  are locally isomorphic;
- 2) for any  $x \in M$  there is an open set  $U$  such that  $\mathcal{F}(U)$  is not empty;
- 3) for any object  $P$  of  $\mathcal{F}(U)$ , the automorphisms of  $P$  are exactly the smooth functions  $U \rightarrow \mathbb{C}^*$ .

*Proposition* [4]: The DD-gerbes are classified by the  $H^3(M, \mathbb{Z})$ .

In analogy with the theory of line bundles there are the notions of connection on the gerbe, but there are two kinds of connections: 0-connections and 1-connections also called by *connective structure* and *curving* respectively in [4, 5]. In [14] the author does not separate the notion of connection to 0 and 1 component, he calls by connection the both structures together.

**2.5. Courant algebroids.** Roughly speaking, Courant algebroid is a bundle with the bracket on it, but the usual conditions on the bracket like antisymmetry, Leibniz rule, Jacobi identity are relaxed in a certain, very special way.

The term Courant algebroid originating from the work [11] was proposed in [18] where the main properties of Courant's bracket (see formula 1.0.1) were axiomatized to the notion of Courant algebroid. (In this paper it was shown that the double of Lie bialgebroid is naturally a Courant algebroid. One can see that naive wish to make a Lie algebroid from the Lie bialgebroid fails and one should use Courant algebroid instead.)

We refer to [24] and references there in for comprehensive study of Courant algebroid.

*Definition:* *Courant algebroid* is a vector bundle  $E$  equipped with a nondegenerate symmetric bilinear form  $\langle, \rangle$ , a bilinear bracket  $[\cdot, \cdot]$  on  $\Gamma(E)$ , and a bundle map  $p : E \rightarrow TM$  satisfying the following properties:

1. The left Jacobi identity  $[e_1, [e_2, e_3]] = [[e_1 e_2], e_3] + [e_2, [e_1, e_3]]$
2. Anchor is homomorphism  $p[e_1, e_2] = [p(e_1), p(e_2)]$
3. Leibniz rule  $[e_1, f e_2] = f[e_1, e_2] + \mathcal{L}_{p(e_1)}(f)e_2$
4.  $[e, e] = \frac{1}{2}\mathcal{D}\langle e, e \rangle$
5. Self-adjointness  $p(e_1)\langle e_2, e_3 \rangle = \langle [e_1, e_2], e_3 \rangle + \langle e_2, [e_1, e_3] \rangle$

where  $\mathcal{D}$  is defined as  $p^*d : C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{p^*} E^* \simeq E$ .

One can also reformulate the definition of Courant algebroid in terms of antisymmetric bracket:

$$\begin{aligned} [e_1, e_2]^{antisym} &= \frac{1}{2}([e_1, e_2] - [e_2, e_1]) \\ [e_1, e_1] &= [e_1, e_2]^{antisym} + \frac{1}{2}\mathcal{D}\langle e_1, e_2 \rangle \end{aligned}$$

Actually this was the point of view accepted in [18], but it is quite easy to reformulate one way to another.

### 3. "CONDUCTING BUNDLE" CONSTRUCTION OF THE COURANT ALGEBROID.

In this section we describe "conducting bundle" construction following ideas of P. Severa. This construction associates Courant algebroid to any extension of bundles of the type  $0 \rightarrow \mathcal{O} \rightarrow \tilde{\mathcal{T}} \xrightarrow{\sigma} \mathcal{T} \rightarrow 0$  in canonical and functorial way. The main property of this construction is that, roughly speaking, the Lie-algebroid structure on such extension gives canonical trivialization of the obtained Courant algebroid. More precisely we require not the arbitrary Lie algebroid structure on this extension, but impose some conditions which are abstracted from the properties of Atiyah algebroids of the line bundles. We call such extensions Picard-Lie-algebroids. This construction can be used to attach Courant algebroid to the gerbe (this will be described in the next section). The intermediate step of the construction is to introduce the "conducting bundle"  $\mathcal{B}_{\tilde{\mathcal{T}}}$ , let us remark that the term "bundle" is a bit misleading because  $\mathcal{B}_{\tilde{\mathcal{T}}}$  does not posses any structure of the  $\mathcal{O}$ -module.

Let us briefly describe the constructions: first step we introduce  $\mathcal{B}_{\tilde{\mathcal{T}}}$  as sheaf endomorphisms of the extension  $0 \rightarrow \mathcal{O} \rightarrow \tilde{\mathcal{T}} \xrightarrow{\sigma} \mathcal{T} \rightarrow 0$  satisfying the certain properties such as Leibniz rule, preservation the extension structure and some other (see subsection 3.1.1), the second step is to construct the Courant algebroid as the factor of the fibered product  $\mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}}$  (see subsection 3.2).

The properties of the construction are roughly speaking the following:  $\mathcal{B}_{\tilde{\mathcal{T}}}$  is an extension of the type:  $0 \rightarrow \Omega^1 \xrightarrow{i} \mathcal{B}_{\tilde{\mathcal{T}}} \xrightarrow{\pi} \mathcal{T} \rightarrow 0$  and it has natural Lie algebra structure; the constructed Courant algebroid is also an extension of the type:  $0 \rightarrow \Omega^1 \rightarrow \mathcal{Q}_{\tilde{\mathcal{T}}} \xrightarrow{\pi_{\mathcal{Q}}} \mathcal{T} \rightarrow 0$ , the bracket and scalar product inherited from the  $\mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}}$  leads precisely to the structure satisfying all axioms for the Courant algebroid.

#### 3.1. Conducting Bundle.

3.1.1. *Definition of the conducting bundle.* For any extension of  $\mathcal{O}$ -modules

$$0 \rightarrow \mathcal{O} \rightarrow \tilde{\mathcal{T}} \xrightarrow{\sigma} \mathcal{T} \rightarrow 0 \quad (3.1.1)$$

one has the sheaf  $\mathcal{B}_{\tilde{\mathcal{T}}}$  (called *conducting bundle*) of  $\mathbb{C}$ -linear endomorphisms of  $\tilde{\mathcal{T}}$  which satisfy properties:

1. it preserves the inclusion  $\mathcal{O} \rightarrow \tilde{\mathcal{T}}$ , and acts on  $\mathcal{O}$  as differentiation, hence there is a map

$$\pi : \mathcal{B}_{\tilde{\mathcal{T}}} \rightarrow \mathcal{T}$$

which associates to a section of  $\mathcal{B}_{\tilde{\mathcal{T}}}$  it's action on  $\mathcal{O}$

2. the endomorphism, which it induces on  $\mathcal{T}$  coincides with the action of the vector field  $\pi(t)$  by Lie derivative;
3. it satisfies the Leibnitz rule (i.e.  $b(fs) = \pi(b)(f)s + fb(s)$ ; where  $b \in \mathcal{B}$ ;  $f \in \mathcal{O}$ ;  $s \in \tilde{\mathcal{T}}$ , in particular it is a differential operator of order one).

*Remark:*  $\mathcal{B}_{\tilde{\mathcal{T}}}$  is not an  $\mathcal{O}$ -module, i.e. the natural structure of  $\mathcal{O}$ -module on  $\mathbb{C}$ -linear endomorphisms of  $\tilde{\mathcal{T}}$  does not respect the subspace  $\mathcal{B}_{\tilde{\mathcal{T}}}$ .



3.1.2. *Motivation for the definition of the conducting bundle.* Suppose that

$$0 \rightarrow \mathcal{O} \rightarrow \tilde{\mathcal{T}} \xrightarrow{\sigma} \mathcal{T} \rightarrow 0$$

is Atiyah algebroid of some linear bundle. For each  $t \in \tilde{\mathcal{T}}$  the endomorphism

$$\text{ad}(t) : s \mapsto [t, s]$$

has the following properties:

1. it preserves the inclusion  $\mathcal{O} \rightarrow \tilde{\mathcal{T}}$
2. the endomorphism it induces on  $\mathcal{O}$  and  $\mathcal{T}$  coincides with the action of the vector field  $\sigma(t)$  by Lie derivative;
3. it satisfies the Leibniz rule (i.e.  $\text{ad}(t)(fs) = \sigma(t)(f)s + f \text{ad}(t)(s)$ ; in particular it is a differential operator of order one).

So one can see that the properties which define  $\mathcal{B}_{\tilde{\mathcal{T}}}$  are precisely the same as the properties of the operators of the adjoint representation of the Atiyah algebroid. So  $\mathcal{B}_{\tilde{\mathcal{T}}}$  is the unification of the all possible operators which can be represented as adjoint representation for the Atiyah Lie-algebroids.

3.1.3. *Inclusion of 1-forms in the conducting bundle.*

*Lemma:* The natural action of  $\Omega^1$  on  $\tilde{\mathcal{T}}$  given by  $\alpha(t) = -\iota_{\sigma(t)}\alpha$  satisfies properties described in the section 3.1.1. Hence, there is a natural map

$$\Omega^1 \xrightarrow{i} \mathcal{B}_{\tilde{\mathcal{T}}}.$$

3.1.4. *The structure and the properties of the conducting bundle.*

*Lemma:*

1. The sequence

$$0 \rightarrow \Omega^1 \xrightarrow{i} \mathcal{B}_{\tilde{\mathcal{T}}} \xrightarrow{\pi} \mathcal{T} \rightarrow 0$$

is exact.

2.  $\mathcal{B}_{\tilde{\mathcal{T}}}$  is closed under the commutator bracket (of endomorphisms of  $\tilde{\mathcal{T}}$ ), hence a sheaf of Lie algebras.
3. The map  $\pi$  is a morphism of sheaves of Lie algebras.
4. The inclusion of  $\Omega^1$  is a morphism of sheaves of Lie algebras, where we consider  $\Omega^1$  as an abelian Lie algebra.
5. The bracket of the elements  $b \in \mathcal{B}_{\tilde{\mathcal{T}}}$  and  $\alpha \in \Omega^1 \subset \mathcal{B}_{\tilde{\mathcal{T}}}$  takes the form:

$$[b, \alpha] = +\mathcal{L}_{\pi(b)}\alpha$$

*Comment:* Sign in the formula above does not depend on whether we define an action of  $\Omega^1$  on  $\tilde{\mathcal{T}}$  by  $\alpha(t) = -\iota_{\sigma(t)}\alpha$  or by  $\alpha(t) = +\iota_{\sigma(t)}\alpha$

3.1.5. *Local description of conducting bundle.* *Remark:* Locally, in terms of any isomorphism  $\tilde{\mathcal{T}} = \mathcal{O} \oplus \mathcal{T}$ , the elements of  $\mathcal{B}_{\tilde{\mathcal{T}}}$  are of the form:

$$\begin{pmatrix} \xi & \alpha \\ 0 & \text{ad}(\xi) \end{pmatrix}$$

with  $\xi \in \mathcal{T}$  and  $\alpha \in \Omega^1$ .

3.1.6. *Automorphisms of extensions.* The category of extensions of  $\mathcal{O}$ -modules of the form

$$0 \rightarrow \mathcal{O} \rightarrow \tilde{\mathcal{T}} \xrightarrow{\sigma} \mathcal{T} \rightarrow 0$$

is a groupoid. The sheaf of local automorphism of an object is canonically isomorphic to  $\Omega^1$  with  $\alpha \in \Omega^1$  acting by  $t \mapsto t - \iota_{\sigma(t)}\alpha$ .

3.1.7. *Automorphisms of conducting bundle.* The category of extensions of the form

$$0 \rightarrow \Omega^1 \rightarrow \mathcal{B} \xrightarrow{\pi_{\mathcal{B}}} \mathcal{T} \rightarrow 0$$

is a groupoid. The sheaf of local automorphisms of an object is canonically isomorphic to  $\underline{\text{Hom}}(\mathcal{T}, \Omega^1)$  with  $\Phi \in \underline{\text{Hom}}(\mathcal{T}, \Omega^1)$  acting by  $b \mapsto b + \Phi(\pi_{\mathcal{B}}(b))$ .

The local automorphisms which respect the Lie-algebra extension structure on  $\mathcal{B}$  are canonically isomorphic morphisms  $\underline{\text{Hom}}(\mathcal{T}, \Omega^1)$  which are  $d^{Lie}$ -closed i.e. are 1-cocycles of the Lie algebra of vector fields with coefficients in  $\Omega^1$  ( this means that:  $L_{\xi}\Phi(\eta) - L_{\eta}\Phi(\xi) = \Phi([\xi, \eta])$ , note that this does not hold for a  $\mathcal{O}$ -linear  $\Phi$ ).

3.1.8. *Functoriality of the conducting bundle.*

*Lemma:* The association  $\tilde{\mathcal{T}} \mapsto \mathcal{B}_{\tilde{\mathcal{T}}}$  is functorial. The induced map  $Aut(\tilde{\mathcal{T}}) \rightarrow Aut(\mathcal{B}_{\tilde{\mathcal{T}}})$  is given by  $\alpha \mapsto (b \mapsto b + [\alpha, b] = b - \mathcal{L}_{\pi(b)}\alpha)$ . Note that  $\mathcal{L}_{\pi(b)}(\alpha)$  is just  $(d^{Lie}\alpha)(\pi(b))$ , hence the corresponding automorphism respects the Lie-algebra extension structure on  $\mathcal{B}$ .

3.2. **Construction of the Courant algebroid.** For any extension  $\tilde{\mathcal{T}}$  there is a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1 & \longrightarrow & \mathcal{B}_{\tilde{\mathcal{T}}} & \longrightarrow & \mathcal{T} & \longrightarrow & 0 \\ & & \text{pr}_{\Omega} \uparrow & & \text{pr}_{\mathcal{B}} \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Omega^1 \oplus \mathcal{O} & \longrightarrow & \mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}} & \longrightarrow & \mathcal{T} & \longrightarrow & 0 \\ & & \text{pr}_{\mathcal{O}} \downarrow & & \text{pr}_{\tilde{\mathcal{T}}} \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \tilde{\mathcal{T}} & \longrightarrow & \mathcal{T} & \longrightarrow & 0 \end{array}$$

Here we denote by  $\mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}}$  the fibered product of  $\mathcal{B}_{\tilde{\mathcal{T}}}$  and  $\tilde{\mathcal{T}}$  with respect to projections on  $\mathcal{T}$ , i.e. subset of  $\mathcal{B}_{\tilde{\mathcal{T}}} \times \tilde{\mathcal{T}}$  such that  $\pi(b) = \sigma(\bar{t})$ .

Let us construct *the Courant algebroid*  $\mathcal{Q}_{\tilde{\mathcal{T}}}$ . We define it as the extension obtained from the middle row by push-out via the map  $\Omega^1 \oplus \mathcal{O} \rightarrow \Omega^1$  given by  $(\alpha, f) \mapsto \alpha - df$ , i.e. we factorize  $\Omega^1 \oplus \mathcal{O}$  and  $\mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}}$  by the elements of the form  $(df, f)$ . Thus,  $\mathcal{Q}_{\tilde{\mathcal{T}}}$  fits into

$$0 \rightarrow \Omega^1 \rightarrow \mathcal{Q}_{\tilde{\mathcal{T}}} \xrightarrow{\pi_{\mathcal{Q}}} \mathcal{T} \rightarrow 0$$

We show below how to introduce the bracket and the scalar product on  $\mathcal{Q}_{\tilde{\mathcal{T}}}$  such that all axioms of Courant algebroid holds true.

*Comment:* We impose  $(df, f) = 0$  (or  $(df, 0) = (0, -f)$ ) with this sign (but not  $(df, -f) = 0$ ) in order to have commutation relation  $[\alpha, q] = (-\mathcal{L}_{\pi(q)}\alpha + \iota_{\pi(q)}(\alpha))$ .

The signs should be agreed in the following two places:

- 1) The natural action of  $\Omega^1$  on  $\tilde{\mathcal{T}}$  given by  $\alpha(t) = -\iota_{\sigma(t)}\alpha$  (see section 3.1.3)
- 2)  $(df, f) = 0$

The result of the agreement is that we have right commutation relation:

$$[\alpha, q] = (-\mathcal{L}_{\pi(q)}\alpha + \iota_{\pi(q)}d(\alpha)) = -d\iota_{\pi(q)}\alpha$$

(see formula 3.2.1)

3.2.1. *The bracket on  $\mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}}$ .* The sheaf  $\mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}}$  is naturally endowed with the bi-linear operation (referred to as “bracket”) defined by the formula

$$[(b_1, t_1), (b_2, t_2)] = ([b_1, b_2], b_1(t_2)) .$$

*Remark:* note that the usual semidirect product Lie bracket  $[(b_1, t_1), (b_2, t_2)] = ([b_1, b_2], b_1(t_2) - b_2(t_1))$  obviously does not preserve the subspace  $\mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}}$ . But described above not antisymmetric bracket is well-defined on this subspace.

*Lemma:* This operation satisfies the left Jacobi identity (i.e. the left adjoint action is by derivations).

This is obvious because  $\mathcal{B}_{\tilde{\mathcal{T}}}$  is a Lie algebra and  $\tilde{\mathcal{T}}$  is a  $\mathcal{B}_{\tilde{\mathcal{T}}}$ -module. The bracket is not skew symmetric, and, thus,  $\mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}}$  is only a Leibniz algebra [20].

3.2.2. *The bracket on  $\mathcal{Q}_{\tilde{\mathcal{T}}}$ .*

*Lemma:* The bracket on  $\mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}}$  descends to  $\mathcal{Q}_{\tilde{\mathcal{T}}}$ , i.e. the subspace  $(df, f)$  is the left and right ideal for the bracket on  $\mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}}$ , (it is obviously not true for the whole space  $\mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}}$ ). For  $q_i \in \mathcal{Q}_{\tilde{\mathcal{T}}}$  and  $\alpha_i \in \Omega^1 \subset \mathcal{B}_{\tilde{\mathcal{T}}}$  the bracket takes the form:

$$[q_1 + \alpha_1, q_2 + \alpha_2] = [q_1, q_2] + \mathcal{L}_{\pi_{\mathcal{Q}}(q_1)}\alpha_2 - \mathcal{L}_{\pi_{\mathcal{Q}}(q_2)}\alpha_1 + d(\iota_{\pi_{\mathcal{Q}}(q_2)}\alpha_1)$$

or it can be rewritten in the form

$$[q_1 + \alpha_1, q_2 + \alpha_2] = [q_1, q_2] + \mathcal{L}_{\pi_{\mathcal{Q}}(q_1)}\alpha_2 - \iota_{\pi_{\mathcal{Q}}(q_2)}d(\alpha_1)$$

So, after the choice of splitting and antisymmetrization the bracket obviously coincides with the original Courant’s bracket 1.0.1.

The proofs are straightforward so let us only comment on the appearance of the additional term  $d(\iota_{\pi_{\mathcal{Q}}(q_1)}\alpha_2)$  in the above formula. It is easy to see from the definitions and item 5 of lemma 3.1.4 that for the  $(\alpha, 0) \in \mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}}$ , where  $\alpha \in \Omega^1 \subset \mathcal{B}_{\tilde{\mathcal{T}}}$  and  $(b, \xi) \in \mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}}$  the bracket takes the form

$$[(\alpha, 0), (b, \xi)] = (-\mathcal{L}_{\pi(b)}\alpha, -\iota_{\sigma(\xi)}(\alpha))$$

In  $\mathcal{Q}_{\tilde{\mathcal{T}}}$  the elements  $(df, 0)$  and  $(0, -f)$  are identified, hence the formula above takes the form:

$$[\alpha, b] = -\mathcal{L}_{\pi(b)}\alpha + d\iota_{\pi(b)}(\alpha) = -\iota_{\pi(b)}(d\alpha) \tag{3.2.1}$$

where we have also used that  $\pi(b) = \sigma(\xi)$ .

*Remark:* the second way to write the formula above emphasize the  $\mathcal{O}$ -linearity in second argument.

*Remark:* Note that

$$[(b, \xi), (\alpha, 0)] = (\mathcal{L}_{\pi(b)}\alpha, 0)$$

without any additional terms.

3.2.3. *The pairing on  $\mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}}$ .* The symmetrized bracket

$$[(b_1, t_1), (b_2, t_2)] + [(b_2, t_2), (b_1, t_1)] = (0, b_1(t_2) + b_2(t_1))$$

leads to the symmetric operation

$$\begin{aligned} \langle , \rangle : \mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}} \times \mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}} &\rightarrow \mathcal{O} \\ ((b_1, t_1), (b_2, t_2)) &\mapsto (b_1(t_2) + b_2(t_1)) \end{aligned}$$

3.2.4. *The pairing on  $\mathcal{Q}_{\tilde{\mathcal{T}}}$  and its properties.*

*Lemma:*

1.  $(\mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}})^{\perp} = \{(df, f) | f \in \mathcal{O}\}$
2.  $\Omega^1 \perp \Omega^1$
3. The pairing descends to a non-degenerate pairing on  $\mathcal{Q}_{\tilde{\mathcal{T}}}$ , i.e. the map

$$\begin{aligned} \mathcal{Q}_{\tilde{\mathcal{T}}} &\rightarrow \underline{\text{Hom}}(\mathcal{Q}_{\tilde{\mathcal{T}}}, \mathcal{O}) \\ q &\mapsto \langle q, \bullet \rangle \end{aligned}$$

is a monomorphism.

4. The diagram

$$\begin{array}{ccc} \mathcal{Q}_{\tilde{\mathcal{T}}} & \longrightarrow & \underline{\text{Hom}}(\mathcal{Q}_{\tilde{\mathcal{T}}}, \mathcal{O}) \\ \uparrow & & \uparrow \pi_{\mathcal{Q}} \\ \Omega^1 & \longrightarrow & \underline{\text{Hom}}_{\mathcal{O}}(\mathcal{T}, \mathcal{O}) \end{array}$$

is commutative, i.e.  $\langle \alpha, q \rangle = \iota_{\pi(q)} \alpha$ .

3.2.5. *Corollary.*

1. The induced bracket on  $\mathcal{Q}_{\tilde{\mathcal{T}}}$  satisfies

$$[q_1, q_2] + [q_2, q_1] = d \langle q_1, q_2 \rangle .$$

2. The inclusion  $\Omega^1 \rightarrow \mathcal{Q}_{\tilde{\mathcal{T}}}$  is the adjoint of the projection  $\mathcal{Q}_{\tilde{\mathcal{T}}} \rightarrow \mathcal{T}$ .

3.2.6. *Functoriality. Lemma:* one can easily see that the association  $\tilde{\mathcal{T}} \mapsto \mathcal{Q}_{\tilde{\mathcal{T}}}$  is functorial. The induced map  $\text{Aut}(\tilde{\mathcal{T}}) \rightarrow \text{Aut}(\mathcal{Q}_{\tilde{\mathcal{T}}})$  is given by  $\alpha \mapsto (q \mapsto q - \iota_{\pi_{\mathcal{Q}}(q)} d\alpha)$ .

3.2.7.  *$\mathcal{Q}_{\tilde{\mathcal{T}}}$  is Courant algebroid.*

*Theorem:*  $\mathcal{Q}_{\tilde{\mathcal{T}}}$  with bracket and the pairing defined above, and the  $\mathcal{O}$ -module structure which will be defined latter is Courant algebroid.

In previous sections we have already established that the bracket and the pairing satisfy all the necessary axioms of Courant algebroids. The only thing to establish is the  $\mathcal{O}$ -module structure and prove Leibniz rule for it. This will be done below (see section 4.1.2). At the moment we do not know natural construction for the  $\mathcal{O}$ -module structure, the reason is that conducting bundle  $\mathcal{B}_{\tilde{\mathcal{T}}}$  does not have an  $\mathcal{O}$ -module structure. So in the next section we will introduce the  $\mathcal{O}$ -module structure in terms of some splitting (called connections there) and we will check that it is independent of the choice of splitting.

3.2.8. *Remark:* in the smooth case any Courant algebroid obtained by such construction is isomorphic to the standard one, non trivial examples can be obtained from gerbes (see section 4.4).

**3.3. Picard-Lie algebroids and trivialization of Courant algebroids constructed from the Picard-Lie algebroids.** The idea of this subsection is, roughly speaking, that Courant algebroids constructed from the extension with brackets are canonically trivialized. More precisely one needs not the arbitrary bracket, but the one with the properties similar to the properties of the bracket on Atiyah algebroids. Such extensions we will call Picard-Lie algebroids.

**3.3.1. Picard-Lie algebroids.** Let us introduce the notion which we call Picard-Lie algebroid, roughly speaking it is Lie algebroid of the type:  $0 \rightarrow \mathcal{O} \rightarrow \tilde{\mathcal{T}} \xrightarrow{\sigma} \mathcal{T} \rightarrow 0$ , with some additional requirement. This definition is just abstractization of the properties of the Atiyah algebroid of some line bundle, relaxing the only requirement of integrality of periods for the Chern class of linear bundle.

*Definition:* The extension  $0 \rightarrow \mathcal{O} \rightarrow \tilde{\mathcal{T}} \xrightarrow{\sigma} \mathcal{T} \rightarrow 0$  is called *Picard-Lie algebroid* (PLA) if:

1.  $\tilde{\mathcal{T}}$  is Lie algebroid with anchor  $\sigma$ ;
2.  $\mathcal{O}$  is abelian ideal in  $\tilde{\mathcal{T}}$ ;
3.  $[\bar{t}, f] = \mathcal{L}_{\sigma(\bar{t})}f$  for  $f \in \mathcal{O} \subset \tilde{\mathcal{T}}$ .

*Example:* Atiyah algebroid of any linear bundle is Picard-Lie algebroid.

*Remark:* obviously all the properties from the section 3.1.2 holds true for the endomorphisms  $ad_{\bar{t}}$  for any Picard-Lie algebroid.

**3.3.2. The construction of the splitting of Courant algebroid coming from PLA.** Suppose that  $\tilde{\mathcal{T}}$  is, in fact, a PLA. The adjoint representation of  $\tilde{\mathcal{T}}$  gives rise to the splitting of  $\text{pr}_{\tilde{\mathcal{T}}}$ , namely  $\bar{t} \mapsto (\text{ad}(\bar{t}), \bar{t})$ .

*Lemma:* on the subspace  $\mathcal{O} \subset \tilde{\mathcal{T}}$  this splitting restricts to the splitting  $f \mapsto (df, f)$  of  $\text{pr}_{\mathcal{O}}$ . (Here we essentially used that  $\mathcal{T}$  is indeed PLA and not arbitrary Lie algebroid).

*Construction of splitting:* consider  $t \in \mathcal{T}$  and take  $\bar{t} \in \tilde{\mathcal{T}}$  such that  $\sigma(\bar{t}) = t$ , define the splitting  $A : \mathcal{T} \rightarrow \mathcal{Q}_{\tilde{\mathcal{T}}}$  by the formula  $t \mapsto (\text{ad}(\bar{t}), \bar{t}) \in \mathcal{Q}_{\tilde{\mathcal{T}}}$ .

*Lemma:* the splitting above does not depend on the choice of  $\bar{t}$ .

*Proof:* the different choices of  $\bar{t}$  differs by some  $f \in \mathcal{O}$ . Due to the previous lemma  $f \mapsto (df, f)$ , by the construction the Courant algebroid is factorization of  $\mathcal{B}_{\tilde{\mathcal{T}}} \times_{\mathcal{T}} \tilde{\mathcal{T}}$  by the elements of the form  $(df, f)$ , hence the splitting is well-defined.

**3.3.3. Properties of the splitting.** *Lemma:* the following holds true:

1. the image of the splitting  $\mathcal{T} \rightarrow \mathcal{Q}_{\tilde{\mathcal{T}}}$  is isotropic with respect to the pairing on  $\mathcal{Q}_{\tilde{\mathcal{T}}}$ ;
2. the image of the splitting  $\mathcal{T} \rightarrow \mathcal{Q}_{\tilde{\mathcal{T}}}$  is closed with respect to the bracket on  $\mathcal{Q}_{\tilde{\mathcal{T}}}$ ;
3. the splitting  $A : \mathcal{T} \rightarrow \mathcal{Q}_{\tilde{\mathcal{T}}}$  is a homomorphism of Lie algebra to the Leibniz algebra.

As a corollary of the all above we obtain.

*Theorem:* Suppose that  $\tilde{\mathcal{T}}$  is a PLA then, the map  $\Omega^1 \oplus \mathcal{T} \rightarrow \mathcal{Q}_{\tilde{\mathcal{T}}}$  given by  $(\alpha, t) \mapsto (\alpha, A(t))$  is an isomorphism of the standard Courant algebroid  $\Omega^1 \oplus \mathcal{T}$  and the Courant algebroid  $\mathcal{Q}_{\tilde{\mathcal{T}}}$ .

## COURANT ALGEBROIDS

PAUL BRESSLER AND ALEXANDER CHERVOV

#### 4. EXACT COURANT

In this section following P. Severa (see also [24] pages 48-50) we show that one can build a kind of differential geometry for Courant algebroids, one can introduce the notion of connection and its curvature. More precisely this can be done for the Courant algebroids of the type:  $0 \rightarrow \Omega^1 \rightarrow \mathcal{Q} \xrightarrow{\pi_{\mathcal{Q}}} \mathcal{T} \rightarrow 0$  which are called *exact* Courant algebroids. But in contrast to the usual situation the curvature form is closed 3-form (not the 2-form as usually) the difference of two connections is 2-form (not the 1-form as usually). Further Severa remarked that, in analogous to the usual situation, adding some 2-form  $\omega$  to some connection one sees that the curvature 3-form changes to the  $d\omega$ , hence we have the well-defined characteristic class of the Courant algebroid (Severa's class). On the other hand starting from the standard Courant algebroid and some closed 3-form Severa proposed to twist the Courant algebroid and to obtain new Courant algebroid.

So one can see that the theory of Courant algebroids is in a sense analogous to the theory of gerbes. This is not accidently and following ideas of P. Severa we show how to attach the Courant algebroid to the gerbe.

In the last subsection we remark that WZNW-Poisson condition introduced recently by Klimcik-Strobl is the same as the condition for graph of the bivector  $\pi$  to be closed under the Courant's bracket in the twisted Courant algebroid constructed by the closed 3-form  $H$ .

At the beginning of the section we pay the debt from the previous section and finish the prove of theorem from the section 3.2.7 that the conducting bundle really gives the Courant algebroid, the only thing to do is to introduce the  $\mathcal{O}$ -module structure on it and to prove the Leibniz rule for it. We do this in this section, because to prove it is more convenient to introduce the notion of connection and to develop some of its properties, after doing this the proof follows easily.

##### 4.1. Exact Courant algebroids.

4.1.1. *Definition.* Let us call Courant algebroid  $\mathcal{Q}$  *exact* Courant algebroid if the following holds true:

1.  $0 \rightarrow \Omega^1 \rightarrow \mathcal{Q} \xrightarrow{\pi_{\mathcal{Q}}} \mathcal{T} \rightarrow 0$ , where  $\pi_{\mathcal{Q}}$  is anchor map;
2.  $\Omega^1$  is an abelian ideal in  $\mathcal{Q}$ ;
3. the pairing satisfy:  $\langle \alpha, q \rangle = \iota_{\pi(q)} \alpha$ .

*Remark:* the Courant algebroid obtained by the conducting bundle construction is obviously exact. Below we will see that up to isomorphism the exact Courant algebroids can be classified by the  $H^3(M, \mathbb{R})$ .

4.1.2. *Alternative description of exact Courant algebroid.* In this section we will give slightly different description of exact Courant algebroid, actually the main thing to prove is that  $\mathcal{O}$ -module structure follows from the other properties. The reason to do this is to pay the debt from the previous section - see corollary below.

*Proposition:* The sheaf of vector spaces  $\mathcal{Q}$  such that the properties below holds true is exact Courant algebroid:

1. there is an exact sequence:

$$0 \rightarrow \Omega^1 \rightarrow \mathcal{Q} \xrightarrow{\pi_{\mathcal{Q}}} \mathcal{T} \rightarrow 0$$

2. there is an bi-linear operation  $[\cdot]$  (referred to as “bracket”) which satisfies left Jacobi identity (i.e. left adjoint action is by differentiation)
3. there is an symmetric bi-linear nondegenerate pairing  $\langle, \rangle: \mathcal{Q} \times \mathcal{Q} \rightarrow \mathcal{O}$
4. the maps in the sequence  $0 \rightarrow \Omega^1 \rightarrow \mathcal{Q} \xrightarrow{\pi_{\mathcal{Q}}} \mathcal{T} \rightarrow 0$  respect the corresponding brackets (we consider  $\Omega^1$  as the abelian Lie algebra)
5. the operations satisfy the following compatibility condition:

$$[q_1, q_2] + [q_2, q_1] = d \langle q_1, q_2 \rangle. \quad (4.1.1)$$

6. The pairing is invariant i.e.

$$\langle [q_1, q_2], q_3 \rangle + \langle q_2, [q_1, q_3] \rangle = \mathcal{L}_{\pi_{\mathcal{Q}}(q_1)} \langle q_2, q_3 \rangle \quad (4.1.2)$$

7. the pairing between the  $\alpha \in \Omega^1 \subset \mathcal{Q}$  and  $q \in \mathcal{Q}$  is given by the formula

$$\langle \alpha, q \rangle = \iota_{\pi_{\mathcal{Q}}(q)} \alpha \quad (4.1.3)$$

8. for  $q \in \mathcal{Q}$  and  $\alpha \in \Omega^1 \subset \mathcal{Q}$  the bracket is given by the formula:

$$[q, \alpha] = \mathcal{L}_{\pi_{\mathcal{Q}}(q)} \alpha. \quad (4.1.4)$$

*Corollary:* this proposition finishes the proof of the main theorem from the previous section (see section 3.2.7, that the conducting bundle construction really gives the Courant algebroid).

The corollary is obvious because we have already established that all the requested in the proposition above properties holds true for the result of the conducting bundle construction, (see lemmas in the sections 3.2.2, 3.2.3, 3.2.5) hence there exist  $\mathcal{O}$ -module structure with the required properties, hence it is Courant algebroid.

4.1.3. The proposition above is actually equivalent to the following lemma.

*Lemma:* Let  $\mathcal{Q}$  satisfy the properties of the proposition above then:

1. There is natural  $\mathcal{O}$ -module structure on  $\mathcal{Q}$  and the following holds:

$$[q_1, f q_2] = f [q_1, q_2] + (\mathcal{L}_{\pi_{\mathcal{Q}}(q_1)} f) q_2 \quad (4.1.5)$$

$$[f q_1, q_2] = f [q_1, q_2] - (\mathcal{L}_{\pi_{\mathcal{Q}}(q_2)} f) q_1 + \langle q_1, q_2 \rangle df \quad (4.1.6)$$

$$\langle f q_1, q_2 \rangle = f \langle q_1, q_2 \rangle \quad (4.1.7)$$

2.  $\Omega^1 \perp \Omega^1$

$$[q_1 + \alpha_1, q_2 + \alpha_2] = [q_1, q_2] + \mathcal{L}_{\pi_{\mathcal{Q}}(q_1)} \alpha_2 - \mathcal{L}_{\pi_{\mathcal{Q}}(q_2)} \alpha_1 + d(\iota_{\pi_{\mathcal{Q}}(q_2)} \alpha_1)$$

4. in the splitted situation  $\mathcal{Q} = \mathcal{T} \oplus \Omega^1$  the pairing is given by the formula

$$\langle (\xi_1, \alpha_1), (\xi_2, \alpha_2) \rangle = \iota_{\xi_1}(\alpha_2) + \iota_{\xi_2}(\alpha_1)$$

In order to see that the properties above holds true we will do, roughly speaking, the following: we introduce the isotropic splitting of the  $\pi_{\mathcal{Q}}(q)$  ( which will be called *connection*) and prove everything with it help, after it can be seen that all the properties does not depend on the choice of connection. To do this some properties of connections should be

mentioned. The  $\mathcal{O}$ -module structure will be introduced in the section 4.2.4 and the proof of the properties will be finished in section 4.2.8.

## 4.2. Connection and its curvature for the Courant algebroid.

4.2.1. *Definition.* Let us call the *connection* on  $\mathcal{Q}$  the map  $A : \mathcal{T} \rightarrow \mathcal{Q}$ , such that it splits projection  $\pi_{\mathcal{Q}}$  i.e.  $\pi_{\mathcal{Q}}A = id$ , (hence the image of  $A$  is transversal to the  $\Omega^1 \subset \mathcal{Q}$ ) and isotropic (i.e.  $\langle A(\xi_1), A(\xi_2) \rangle = 0$ ).

4.2.2. *Remark.* The space of connections is affine space under the vector space of  $\Omega^2$ . The connection  $A + \omega$  is given by the map  $A + \omega : \xi \mapsto A(\xi) + \iota_{\xi}\omega$ . One can easily see from the formula 4.1.3 that the new map will also be isotropic and it's obviously splits the map  $\pi_{\mathcal{Q}}$  i.e.  $\pi_{\mathcal{Q}}(A + \omega) = id$ .

4.2.3. *Lemma:* All connections can be obtained from the given one adding to it some 2-form  $\omega$ .

Consider two connections  $A_1$  and  $A_2$ . Let  $\alpha_{\xi} = A_1(\xi) - A_2(\xi)$ . From transversality follows that  $\alpha_{\xi}$  is 1-form for any given  $\xi$ . Using isotropicity one obtains:  $0 = \langle A_1(\xi_1) + \alpha_{\xi_1}, A_1(\xi_2) + \alpha_{\xi_2} \rangle$  hence  $\iota_{\xi_2}\alpha_{\xi_1} = \iota_{\xi_1}\alpha_{\xi_2}$  hence  $\alpha_{\xi} = \iota_{\xi}\omega$  for some 2-form  $\omega$ .

4.2.4. *Construction of the  $\mathcal{O}$ -module structure on  $\mathcal{Q}$ .* Let us choose any connection  $A$ . Any element  $q \in \mathcal{Q}$  can be represented uniquely as  $q = \alpha + A(\xi)$ , for some  $\alpha$  and  $\xi$ . Let us define

$$fq \stackrel{def}{=} A(f\xi) + f\alpha$$

Obviously such multiplication by functions commutes with the change of connection given by  $A \rightarrow A + \omega$  i.e.  $f((A + \omega)(\xi)) = (A + \omega)(f\xi)$  hence in view of lemma 4.2.3 the multiplication by functions does not depend on the choice of connection.

4.2.5. *Definition:* let us call the *curvature* of the connection  $A$  the map  $F : \mathcal{T} \times \mathcal{T} \rightarrow \Omega^1$  given by  $F(x, y) = [A(x), A(y)] - A([x, y])$ . Due to the isotropicity of connection and the formula 4.1.1 this map is antisymmetric in  $x$  and  $y$ .

Let us call the 3-form  $H(\xi_1, \xi_2, \xi_3)$  defined by the formula  $H(\xi_1, \xi_2, \xi_3) = \iota_{\xi_3}F(\xi_1, \xi_2)$  the *curvature 3-form* for the connection  $A$ .

4.2.6. *Lemma.* One can obviously see from the isotropicity of connection and formula 4.1.3 that

$$H(\xi_1, \xi_2, \xi_3) = \langle [A(\xi_1), A(\xi_2)], A(\xi_3) \rangle \quad (4.2.1)$$

4.2.7. *Lemma.* Curvature 3-form satisfies the following properties: it is antisymmetric in all three arguments, it is  $\mathcal{O}$ -linear and it is closed.

To argue these properties one goes as follows. By the invariance of the pairing ( formula 4.1.2) and isotropicity of connection one has:  $\langle [A(\xi_1), A(\xi_2)], A(\xi_3) \rangle = - \langle [A(\xi_1), A(\xi_3)], A(\xi_2) \rangle$ . Hence curvature 3-form is antisymmetric in all three arguments.

It is obviously  $\mathcal{O}$ -linear in the third argument, hence by antisymmetry it is  $\mathcal{O}$ -linear in all three arguments.

Closedness (Bianchi identity) follows from the Jacobi identity as usually :

$$Jac(A(\xi_1), A(\xi_2), A(\xi_3)) = -\iota_{\xi_1}\iota_{\xi_2}\iota_{\xi_3}dH$$



where  $Jac(q_1, q_2, q_3)$  is "Jacobiator" given by the formula  $[q_1, [q_2, q_3]] - [[q_1, q_2], q_3] - [q_2, [q_1, q_3]]$

To check this formula it is convenient to use the formula A.2, it is obvious that the above reasoning is analogous to the one in section 4.3.2.

4.2.8. *Proof of the formulas 4.1.5, 4.1.6, 4.1.7.* The formula 4.1.7 obviously follows from the formula 4.1.3 and isotropicity of connection. To prove 4.1.5 one should note that it's holds for the case  $[q, f\alpha]$  (in view of the formula 4.1.3). To check it for  $[A(\xi_1), fA(\xi_2)]$  one proceeds as follows:  $[A(\xi_1), fA(\xi_2)] = A[\xi_1, f\xi_2] + F(\xi_1, f\xi_2) = fA[\xi_1, \xi_2] + (\mathcal{L}_{\xi_1}f)A(\xi_2) + fF(\xi_1, \xi_2) = f[A(\xi_1), A(\xi_2)] + (\mathcal{L}_{\xi_1}f)A(\xi_2)$ . We have used the  $\mathcal{O}$ -linearity of the curvature  $F$ .

So the formula 4.1.5 is proved. The formula 4.1.6 follows from it and the formula 4.1.1.

**4.3. Further properties of exact Courant algebroids.** We will study isomorphisms, deformations and classification of exact Courant algebroids.

4.3.1. *Isomorphisms. Lemma:* The sheaf of local isomorphisms of  $\mathcal{Q}$  is canonically isomorphic to the sheaf of closed 2-forms  $\Omega^{2,cl}$  the action is given by the formula  $\Phi_\omega : q \mapsto q + \iota_{\pi_{\mathcal{Q}}(q)}\omega$ .

This can be seen from the formula:

$$[\Phi_\omega(q_1), \Phi_\omega(q_2)] = \Phi_\omega[q_1, q_2] - \iota_{\pi_{\mathcal{Q}}(q_1)}\iota_{\pi_{\mathcal{Q}}(q_2)}d\omega \quad (4.3.1)$$

This formula follows from A.1.

4.3.2. *Twisting the Courant algebroid by the closed 3-form.*

*Lemma:* The sheaf of local deformations of the bracket on  $\mathcal{Q}$  is canonically isomorphic to the sheaf of closed 3-forms  $\Omega^{3,cl}$  the corresponding deformation is given by the formula:  $[q_1, q_2]_{new} = [q_1, q_2] + \iota_{\pi_{\mathcal{Q}}(q_1)}\iota_{\pi_{\mathcal{Q}}(q_2)}H$ .

This can be seen from the formula:

$$Jac(q_1, q_2, q_3)_{new} = \iota_{\pi_{\mathcal{Q}}(q_1)}\iota_{\pi_{\mathcal{Q}}(q_2)}\iota_{\pi_{\mathcal{Q}}(q_3)}dH$$

Where  $Jac(q_1, q_2, q_3)_{new}$  is "Jacobiator" with respect to the new bracket, i.e.  $Jac(q_1, q_2, q_3)_{new} = [q_1, [q_2, q_3]_{new}]_{new} - [[q_1, q_2]_{new}, q_3]_{new} - [q_2, [q_1, q_3]_{new}]_{new}$

Note that such deformations preserve the pairing.

To check this formula it is convenient to use the formula A.2, it is obvious that the above reasoning is analogous to the one in section 4.2.7.

4.3.3. *Lemma.* The deformations which correspond to exact 3-forms  $H = d\omega$  are trivial, the isomorphism is given by the expected rule:  $q \mapsto q + \iota_{\pi_{\mathcal{Q}}(q)}\omega$ . This can be seen from the formula 4.3.1

4.3.4. *Lemma.* The curvature of the connection  $A + \omega$  equals to  $H(A) + d\omega$ .

This statement is essentially equivalent to the formula 4.3.1.

4.3.5. *Characteristic class of exact Courant algebroid. (Severa's class).* So we have seen that taking the curvature 3-form of any connection one obtains closed 3-form, we have also seen that taking another connection curvature changes up to exact 3-form.

So for the given exact Courant algebroid there is well-defined cohomology class  $[H] \in H^3(M)$ .

*Example:* if we take standard Courant algebroid and the connection  $\xi \mapsto (\xi, 0) \in \mathcal{Q}$ , then the curvature of such connection is zero. If we take the Courant algebroid obtained by deformation of standard one by 3-form  $\Omega$  (see section 4.3.2) and the same connection then it's curvature 3-form equals to  $-\Omega$ .

4.3.6. *Addition.* For the extensions  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$ :  $0 \rightarrow \Omega^1 \rightarrow \mathcal{Q}_i \xrightarrow{\pi} \mathcal{T} \rightarrow 0$  one can define their Baer sum. Let us remark that if  $\mathcal{Q}_i$  are Courant algebroids then there is natural Courant algebroid structure on the Baer sum of  $\mathcal{Q}_i$ . The characteristic class of it equals to the sum of characteristic classes.

To check this one proceeds as follows: consider the fibered product of  $\bar{\mathcal{Q}} \stackrel{def}{=} \mathcal{Q}_1 \times_{\mathcal{T}} \mathcal{Q}_2$  by definition the Baer sum  $\mathcal{Q}_{1+2}$  is factor of  $\bar{\mathcal{Q}}$  by the skewdiagonal  $(\alpha, -\alpha)$  where  $\alpha \in \Omega^1 \subset \mathcal{Q}_i$ , with the embedding of  $\Omega^1 \rightarrow \mathcal{Q}_{1+2}$  given by  $\alpha \rightarrow (\alpha, 0)$ . So one has  $0 \rightarrow \Omega^1 \rightarrow \mathcal{Q}_{1+2} \xrightarrow{\pi} \mathcal{T} \rightarrow 0$ . The bracket and the pairing on  $\bar{\mathcal{Q}}$  are induced from the imbedding  $\bar{\mathcal{Q}} \subset \mathcal{Q}_1 \times \mathcal{Q}_2$ , one can easily see that  $\bar{\mathcal{Q}}$  is subalgebra in  $\mathcal{Q}_1 \times \mathcal{Q}_2$ . To endow with the bracket and pairing the  $\mathcal{Q}_{1+2}$  one easily checks that skewdiagonal  $(\alpha, -\alpha)$  is both-sided ideal for the bracket and orthogonal to  $\bar{\mathcal{Q}}$ . So both the pairing and the bracket can be defined on  $\mathcal{Q}_{1+2}$ .

4.3.7. *Generalization.* (We are grateful for A. Kotov for suggestive discussion on this point). All the statements of this section are straightforwardly true for any Courant algebroid of the form:

$$0 \rightarrow \mathcal{A}^* \rightarrow \mathcal{Q} \xrightarrow{\pi_{\mathcal{Q}}} \mathcal{A} \rightarrow 0$$

where  $\mathcal{A}$  is any algebroid and  $\mathcal{Q}$  is defined by the axioms 4.1.2. We note that in this situation there is natural map  $\Omega^1 \rightarrow \mathcal{A}^*$  which is dual to the anchor  $\mathcal{A} \rightarrow \mathcal{T}$ , hence formulas like 4.1.1 make sense. All propositions holds true in this situation because all the formulas of differential calculus: Lie derivatives, deRham complex,  $\iota_a$  etc. holds identically true for any Lie algebroid  $\mathcal{A}$  not only for  $\mathcal{T}$  (see [21] or for example section 2 of [13]). For example such Courant algebroids are classified by the  $H^3(M, \mathcal{A})$  - Lie algebroid cohomology of  $\mathcal{A}$ .

4.4. **Courant algebroid of gerbe.** Both DD-gerbes and exact Courant algebroids are classified by the third cohomology group, so one can just take the class of the gerbe and construct the exact Courant algebroid with the same class, but one can hope there is more direct relation. As we already mentioned P. Severa proposed that the role of the Courant algebroid is an Atiyah algebroid of a gerbe, but at the moment not everything is clear with this analogy. Let us sketch the construction (idea of which is due to Severa's) of Courant algebroid from gerbes. Construction consists of three step.

Consider gerbe  $F$  on  $M$ .

**Step 1.** Gerbes are classified by  $H^3(X, \mathbb{Z}) = H^2(X, \mathbb{C}^*)$ , where following Brylinski we denoted by  $\underline{\mathbb{C}^*}$  the sheaf of invertible functions. Let us choose the Čech representative of the characteristic class of gerbe, it can be given by some invertible function  $g_{\alpha\beta\gamma}$  on the

intersection of charts  $U_{\alpha\beta\gamma}$ . Let us choose the 1-forms  $A_{\alpha\beta}$  on each intersection of charts  $U_{\alpha\beta}$ , such that they satisfy the condition:

$$A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = d\log(g_{\alpha\beta\gamma}) \quad (4.4.1)$$

This can be always done for the small enough covering  $U_\alpha$ , because  $g_{\alpha\beta\gamma}$  is Čech cocycle. As it is explained in [5] such choice is essentially the choice of what is called "0-connection on gerbe" or "connective structure".

**Step 2.** Let us consider on each chart  $U_\alpha$  bundles  $B_\alpha = \mathbb{C}U_\alpha \oplus \mathcal{T}U_\alpha$ . On the intersection of charts  $U_{\alpha\beta}$  consider the maps  $\Phi_{\alpha\beta} : B_\alpha \rightarrow B_\beta$  given by the 1-forms  $A_{\alpha\beta}$  by the rule:  $\Phi_{\alpha\beta} : (f, \xi) \mapsto (f + \iota_\xi A_{\alpha\beta}, \xi)$ . Obviously one cannot glue together  $B_\alpha$  to the global object because the triple product  $\Phi_{\alpha\beta}\Phi_{\beta\gamma}\Phi_{\gamma\alpha}$  is not identity, but given by the closed 1-form  $d\log(g_{\alpha\beta\gamma})$ .

**Step 3.** Let us apply the conducting bundle construction to each  $B_\alpha$ . We obtain Courant algebroid  $C(B_\alpha)$  on each chart  $U_\alpha$ , by the functoriality of the construction we have the maps  $C(\Phi_{\alpha\beta}) : C(B_\alpha) \rightarrow C(B_\beta)$  on the intersection of charts  $U_{\alpha\beta}$ .

The main point is that the triple product  $C(\Phi_{\alpha\beta})C(\Phi_{\beta\gamma})C(\Phi_{\gamma\alpha})$  does equals the identity, because the automorphism of extension  $B_\alpha$  given by the 1-form  $\omega$  induces automorphism of the Courant algebroid given by 2-form  $d\omega$ , hence  $C(\Phi_{\alpha\beta})C(\Phi_{\beta\gamma})C(\Phi_{\gamma\alpha}) = Id + d(d\log(g_{\alpha\beta\gamma})) = Id$ . Hence we have the globally defined exact Courant algebroid. One can obviously see that its characteristic class coincides with the characteristic class of the gerbe, more precisely with its image in  $H^3(\mathbb{Z}) \otimes \mathbb{R}$ .

The construction is finished.

Let us make few comments on the construction. First one can see that the construction is not direct: we work with the characteristic class of gerbe, but not with the sheaf of groupoids itself, it would be very interesting if one can propose the construction analogous to Atiyah algebroid construction which works with the sheaf of groupoid itself and makes precise sense of the "groupoid of its symmetries" and "Courant algebroid of its infinitesimal symmetries". Second P. Severa remarked that the choice of 1-connection on gerbe leads to connection on Courant algebroid (in a sense explained in section 4) this is in complete analogy with line bundle - Atiyah algebroid situation, where the connection is the same as the splitting of Atiyah algebroid. Third P. Severa remarked that bundles  $B_\alpha$  considered in step 3 of the construction should be understood as connective structure functor (see [4, 5]) applied to the object of groupoid on  $U_\alpha$  (there is only one object up to isomorphism for the small enough  $U_\alpha$ ). The map  $B_\alpha \rightarrow B_\beta$  should arise (after the choice of trivialization of  $A(L_{\alpha\beta})$ ) from the map  $B_\alpha \rightarrow B_\beta \oplus_{Baer} A(L_{\alpha\beta})$  which should arise from the some functoriality properties, where  $L_{\alpha\beta}$  is the line bundle on the intersection of charts which is part "gerbe data" description of gerbes. This picture is very interesting because it works directly with the sheaf of groupoids, not only with the characteristic classes of gerbes, but at the moment not all the details clear for us how does the map  $B_\alpha \rightarrow B_\beta \oplus_{Baer} A(L_{\alpha\beta})$  arises.

**4.5. Courant algebroids and WZNW-Poisson manifolds.** In this section we made a remark that WZNW-Poisson condition of Klimcik and Strobl [16] (formula 4.5.1 below) coincides with the condition that graph of bivector  $\pi$  is subalgebra in twisted Courant algebroid (see Lemma 4.5.3 below).

In the paper [16] the authors considered generalization of Poisson  $\sigma$  - model with the help of the WZNW-type term given by the three form  $H$ . They found the consistency condition which should be imposed on the bivector  $\pi$  and closed three form  $H$ . Their condition is the following:

$$[\pi, \pi]_{Schouten} = 2 \langle H, \pi \otimes \pi \otimes \pi \rangle \quad (4.5.1)$$

(We denote by  $\langle, \rangle$  the contraction of polyvectors and forms).

Contraction in  $\langle H, \pi \otimes \pi \otimes \pi \rangle$  is given with respect to the first, third and fifth entry.

4.5.1. *Example (due to N. Markarian):* let us take some nondegenerate, but possibly not closed 2-form  $\omega$  and let  $H = d\omega$ . Take  $\pi = \omega^{(-1)}$  then the pair  $\pi, H$  obviously satisfies WZNW-Poisson condition.

4.5.2. Let us denote by the  $\mathcal{Q}_H$  the exact Courant algebroid obtained from the standard one by twisting the bracket by the three form  $H$ , see subsection 4.3.2.

Here we observe the following:

4.5.3. *Lemma.* The graph of the mapping  $\Omega^1 \rightarrow \mathcal{T}$  given by the bivector  $\pi$  by the rule  $\alpha \mapsto \langle \pi, \alpha \rangle$  is *subalgebra* in the exact Courant algebroid  $\mathcal{Q}_H$  if and only if the WZNW-Poisson equation 4.5.1 is satisfied.

This lemma is obviously generalization of the original Courant's theorem which states that Poisson bracket can be characterized by the property that its graph is subalgebra in standard Courant algebroid.

The lemma is equivalent to the following formula:

$$\begin{aligned} [\langle \pi, \alpha \rangle, \langle \pi, \beta \rangle] = & \langle [\pi, \pi], \alpha \wedge \beta \rangle + \langle \pi, \mathcal{L}_{\langle \pi, \alpha \rangle} \beta \rangle - \\ & - \langle \pi, \mathcal{L}_{\langle \pi, \beta \rangle} \alpha \rangle - \langle \pi, d \langle \pi, \alpha \wedge \beta \rangle \rangle \end{aligned} \quad (4.5.2)$$

This formula can be obtained applying two times the formula A.4 to the LHS ( see example A.5), then one remarks that RHS can be transformed by the formula  $\langle \pi, \mathcal{L}_{\langle \pi, \alpha \rangle} \beta \rangle - d \langle \pi, \alpha \wedge \beta \rangle = \langle \langle \pi, \alpha \rangle, d\beta \rangle$  and finally note that:  $\langle \pi, \alpha \rangle \langle \pi, d\beta \rangle = \langle \pi, \langle \langle \pi, \alpha \rangle, d\beta \rangle \rangle$ .

4.5.4. *Remark:* The antisymmetric bracket in standard Courant algebroid on the graph of Poisson bivector  $\pi$  can be rewritten as follows:

$$[\langle \pi, \alpha \rangle, \langle \pi, \beta \rangle]_{Cour} = ([\langle \pi, \alpha \rangle, \langle \pi, \beta \rangle]_{vect}, [\alpha, \beta]_{Poisson})$$

Where  $[\cdot, \cdot]_{Poisson}$  - is the usual bracket on 1-forms [12] defined by the rules  $[df, dg] = d\{f, g\}$ , and Leibniz rule for functions  $[hdf, dg] = \{h, g\}df + h[df, dg]$ ;  $[f, dg] = \{f, g\}$ . which can be rewritten explicitly as follows:  $[\alpha, \beta] = \mathcal{L}_{\langle \pi, \alpha \rangle} \beta - \mathcal{L}_{\langle \pi, \beta \rangle} \alpha - d \langle \pi, \alpha \wedge \beta \rangle$ .

*Remark:* The previous remark can be reformulated saying that the map  $\pi : \Omega^1 \rightarrow \mathcal{Q}$ , given by  $\alpha \mapsto (\langle \pi, \alpha \rangle, \alpha)$  respects the brackets i.e.  $[\cdot, \cdot]_{Poisson}$  on  $\Omega^1$  and antisymmetric Courant's bracket on  $\mathcal{Q}$ . This essentially means that:  $[\langle \pi, \alpha \rangle, \langle \pi, \beta \rangle]_{vect} = \langle \pi, [\alpha, \beta]_{Poisson} \rangle$

## 5. ALGEBROIDS ON THE LOOP SPACES AND POISSON $\sigma$ -MODEL.

This section originated from the reading the papers [16, 17] related to Poisson  $\sigma$ -model and to some extent the papers [22, 23]. We will try to give more geometric and coordinateless reformulation of some of their formulas.

We remark that the phase space of Poisson  $\sigma$ -model is just cotangent bundle to the loop space of the manifold. This may be interesting for the geometrically oriented reader, because the coordinate description may seem to be quite artificial and this coordinateless description seems to be very natural.

In the papers [17] and in some sense in [16] some algebroids played an important role. These algebroids were constructed from the physical reasons not completely clear to us, they played role of symmetries in Poisson  $\sigma$ -models. We propose the construction that starting from the Lie algebroid on the manifold one can construct Lie algebroid on the loop space of the manifold. (The construction does not work for the Courant algebroids. Courant algebroid should give Picard-Lie algebroid on the loop space, which is obvious on the level of the characteristic classes, but at the moment we do not know the direct construction). We remark that after the choice of coordinates, considering the cotangent bundle of Poisson manifold as Lie algebroid the constructed algebroid on the loop space is very similar to the algebroid considered in [17]. The natural generalization of this is to consider the Lie algebroid which originates from the Dirac structures in the Courant algebroid, we hope that considering the Dirac structure coming from WZNW-Poisson condition (see section 4.5) and applying to it our construction one obtains the same Lie algebra as in [16].

This section organized as follows first we explain the construction of Lie algebroid on the loop space, starting from the one on the manifold itself, after we give geometric reformulation of Poisson  $\sigma$ -model and explain the relation of the first construction to the formulas from the paper [17].

### 5.1. From Lie-algebroid on the manifold to the Lie-algebroid on the loop space.

We start from the recollection of some well-known facts about the loop spaces and more generally about the space of maps from one manifold to another. We work with smooth manifolds, although most everything extends to the algebraic situation.

5.1.1. The tangent bundle to the  $Map(X, Y)$  equals to the maps into tangent bundle  $Map(X, TY)$ . Note that it is obviously not true for the cotangent bundle:  $Map(X, T^*Y)$  NOT equal to  $T^*Map(X, Y)$ . Also  $\Lambda^n TMap(X, Y)$  not equal to  $TMap(X, \Lambda^n Y)$ .

5.1.2. There is an obvious map from the vector fields on  $Y$  to the vector fields on  $Map(X, Y)$ , geometrically this means that we can move the image of  $X$  by the flow generated by vector field  $\xi$  on  $Y$ . Or this can be said formally that considering the composition of  $f : X \rightarrow Y$  and  $\xi : Y \rightarrow TY$  we obtain the map  $X \rightarrow TY$  and hence the element of the  $TMap(X, Y)$ .

*Lemma:* this correspondence is obviously homomorphism of Lie algebras.

5.1.3. There is obviously map  $\mathbb{C} \times TY \rightarrow TY$ , (just the multiplication of vector fields by scalars) hence one has map  $Map(X, \mathbb{C}) \times Map(X, TY) \rightarrow Map(X, TY)$ . We will denote this multiplication of vector fields on  $Map(X, Y)$  by functions on  $X$  as  $f(x) * \xi$ , in order to distinct it from the multiplication by functions on  $Map(X, Y)$ .

*Lemma:* for the vector fields  $\xi_i$  on  $Map(X, Y)$  which came from the vector fields on  $Y$  the following holds:  $[f_1(x) * \xi_1, f_2(x) * \xi_2] = (f_1(x)f_2(x)) * [\xi_1, \xi_2]$ , where  $f_i$  are functions on  $X$  and the multiplication is described above.

*Remark:* the lemma above is obviously false without assumption that  $\xi_i$  came from the vector fields on  $Y$ .

The lemma is analogous to the following:  $[\sum_i a_i f_i(t_i) \frac{\partial}{\partial t_i}, \sum_j b_j h_j(t_j) \frac{\partial}{\partial t_j}] = \sum_i a_i b_i [f_i(t_i) \frac{\partial}{\partial t_i}, h_i(t_i) \frac{\partial}{\partial t_i}]$  for the vector fields  $\sum_i f_i(t_i) \frac{\partial}{\partial t_i}$  and  $\sum_i h_i(t_i) \frac{\partial}{\partial t_i}$  where function  $f_i$  and  $h_i$  depends only on  $t_i$ .

*Remark:* informally this can be expressed as follows: the vector fields on  $Map(X, Y)$  which came from the vector fields on  $Y$  can be written as  $\int_X F(Y(x)) \frac{\delta}{\delta Y(x)}$ . In complete analogy with  $\sum_i f(Y_i) \frac{\partial}{\partial Y_i}$  but the index  $i$  is substituted by the continuous index  $x$ .

*Remark:* this multiplication by function on  $X$  does not respect the subalgebra of vector fields which came from  $Y$ .

*Remark:* another way to describe such multiplication comes from lemma 5.1.4, one will see that  $TMap(X, Y) = pr_* ev^* TM$  and  $ev^* TM$  obviously has structure of  $\mathcal{O}(X)$  module.

5.1.4. *Lemma:* Consider the diagram:

$$\begin{array}{ccc} Map(X, Y) \times X & \xrightarrow{ev} & Y \\ \text{pr} \downarrow & & \text{pr}_X \downarrow \\ Map(X, Y) & & X \end{array} \quad (5.1.1)$$

where  $ev$  - is the evaluation map.

Then the following holds true (in the next subsection we will say the same more explicitly):

1.  $ev^* TX$  has natural structure of Lie algebra (pay attention that here we used inverse image as a bundle i.e. as sheaf of  $\mathcal{O}$ -modules, this wrong-way functoriality for the vector fields is due to the map  $TY \rightarrow TMap(X, Y)$ ).
2. One can see that  $pr_* ev^* TX = TMap(X, Y)$  (this is obviously not true if one puts any other space instead of  $Map(X, Y)$  in the diagram 5.1.1 this property is characteristic for the space  $Map(X, Y)$ ).
3. The map  $pr^*$  is isomorphism of Lie algebras  $ev^* TX \rightarrow TMap(X, Y)$ .
4.  $pr_*(ev^* T^* Y \otimes pr_X^* \Omega^d TX) = T^* Map(X, Y)$ , where  $\Omega^d TX$  - highest forms on  $X$  (more exactly one should consider measures on  $X$ , but let us do not go into such details).

5.1.5. *Explicit description of Lie algebra structure on  $ev^* TY$  and its action on  $\mathcal{O}(LM)$ .* By definition  $ev^* TY$  is  $\mathcal{O}(Map(X, Y) \times Y) \otimes_{\mathcal{O}(Y)} ev^{-1} TY$  hence its elements can be written as

$\left( F(\gamma) \otimes_{\mathbb{C}} h(x) \right) \otimes_{\mathcal{O}(Y)} \xi$ , where  $\xi$  is vector field on  $Y$ , and  $\gamma \in Map(X, Y)$ , and  $F, h$  are functions on  $Map(X, Y)$  and  $X$  respectively.

Let us write the formulas corresponding to the lemma above:

1. The item 3 in previous lemma stated the isomorphism  $ev^* TX \rightarrow TMap(X, Y)$ . So we should explain the action of  $\left( F(\gamma) \otimes_{\mathbb{C}} h(x) \right) \otimes_{\mathcal{O}(Y)} \xi$  on functions on  $Map(X, Y)$ .

$$\left( \left( F(\gamma) \otimes_{\mathbb{C}} h(x) \right) \otimes_{\mathcal{O}(Y)} \xi \right) \Phi(\gamma) = F(\gamma) ((h(x) * \xi) [\Phi(\gamma)]) \quad (5.1.2)$$

one multiplies  $(h(x) * \xi)$  as described in subsection 5.1.3

2. The multiplication of vector fields on  $Map(X, Y)$  on functions  $f(x) \in \mathcal{O}(X)$  (see section 5.1.3) in this notations is obviously written as:

$$f(x) * \left( \left( F(\gamma) \otimes_{\mathbb{C}} h(x) \right) \otimes_{\mathcal{O}(Y)} \xi \right) = \left( F(\gamma) \otimes_{\mathbb{C}} f(x)h(x) \right) \otimes_{\mathcal{O}(Y)} \xi \quad (5.1.3)$$

3. The commutator on  $ev^*TY$  can be written explicitly as follows

$$\begin{aligned} & \left[ \left( F_1(\gamma) \otimes_{\mathbb{C}} h_1(x) \right) \otimes_{\mathcal{O}(Y)} \xi_1, \left( F_2(\gamma) \otimes_{\mathbb{C}} h_2(x) \right) \otimes_{\mathcal{O}(Y)} \xi_2 \right] = \\ & \quad \left( F_1(\gamma)F_2(\gamma) \otimes_{\mathbb{C}} h_1(x)h_2(x) \right) \otimes_{\mathcal{O}(Y)} [\xi_1, \xi_2] + \\ & \quad \left( F_1(\gamma)((h_1 * \xi_1)[F_2](\gamma)) \otimes_{\mathbb{C}} h_2(x) \right) \otimes_{\mathcal{O}(Y)} \xi_2 - \\ & \quad \left( F_2(\gamma)((h_2 * \xi_2)[F_1](\gamma)) \otimes_{\mathbb{C}} h_1(x) \right) \otimes_{\mathcal{O}(Y)} \xi_1 \end{aligned} \quad (5.1.4)$$

The expression above is well-defined with respect to the tensor product taken over  $\mathcal{O}(Y)$ .

*Remark:* the formula 5.1.4 is quite obvious the only worth noting remark is that looking on the LHS it may be tempting to write the wrong expression in the RHS:

$$\left( F_1(\gamma)((\xi_1)[F_2](\gamma)) \otimes_{\mathbb{C}} h_1h_2(x) \right) \otimes_{\mathcal{O}(Y)} \xi_2$$

The simple form of the first term in LHS is due to the lemma 5.1.3.

*Remark:* one can prove the formula 5.1.4 as follows: first

$$[(1 \otimes_{\mathbb{C}} f_1(z)) \otimes_{\mathcal{O}(Y)} \xi_1, (1 \otimes_{\mathbb{C}} f_2(z)) \otimes_{\mathcal{O}(Y)} \xi_2] = (1 \otimes_{\mathbb{C}} f_1(x)f_2(x)) \otimes_{\mathcal{O}(Y)} [\xi_1, \xi_2] \quad (5.1.5)$$

this is just the reformulation of the lemma in section 5.1.3. The rest part of the formula follows from the Leibniz rule with respect to the multiplication on function  $F(\lambda) \in \mathcal{O}(Map(X, Y))$  and formula 5.1.2 of the explicit description of the actions of vector fields on  $Map(X, Y)$  on functions on  $Map(X, Y)$ .

4. The elements of  $(ev^*T^*Y \otimes pr_X^*\Omega^dTX)$  can be written as

$$(F(\gamma) \otimes_{\mathbb{C}} \omega(x)) \otimes_{\mathcal{O}(Y)} \alpha(y).$$

The pairing between  $T^*Map(X, Y) = pr_*(ev^*T^*Y \otimes pr_X^*\Omega^dTX)$  and  $TMap(X, Y)$  can be given explicitly as

$$\begin{aligned} & \langle (F_1(\gamma) \otimes_{\mathbb{C}} \omega(x)) \otimes_{\mathcal{O}(Y)} \alpha(y); (F_2(\gamma) \otimes_{\mathbb{C}} f(x)) \otimes_{\mathcal{O}(Y)} \xi(y) \rangle = \\ & = F_1(\gamma)F_2(\gamma) \int_X (f(x) \langle \alpha; \xi \rangle_{\gamma(X)} \omega(x)). \end{aligned}$$

5.  $TMap(X, Y)$  acts by Lie derivatives on  $T^*Map(X, Y)$ , the action can be described explicitly as follows:

$$\begin{aligned} & \left( F_1(\gamma) \otimes_{\mathbb{C}} h(x) \right) \otimes_{\mathcal{O}(Y)} \xi_1 \left[ \left( F_2(\gamma) \otimes_{\mathbb{C}} \mu(x) \right) \otimes_{\mathcal{O}(Y)} \alpha \right] = \\ & F_1((h * \xi)F_2) \otimes_{\mathbb{C}} \mu(x) \otimes_{\mathcal{O}(Y)} \alpha + F_1F_2 \otimes_{\mathbb{C}} h(x)\mu(x) \otimes_{\mathcal{O}(Y)} \xi[\alpha] \end{aligned} \quad (5.1.6)$$

5.1.6.  $pr_*ev^*$  of an algebroid is an algebroid. Consider the diagram:

$$\begin{array}{ccc} Map(X, Y) \times X & \xrightarrow{ev} & Y \\ pr \downarrow & & pr_X \downarrow \\ Map(X, Y) & \times & X \end{array} \quad (5.1.7)$$

where  $ev$  - is the evaluation map.

*Lemma:* Let  $A$  be an Lie algebroid on  $Y$ , with the anchor  $p : A \rightarrow \mathcal{T}_Y$ , then there is well-defined bracket on  $ev^*A$ , which satisfy Jacobi identity given by the formula:

$$\begin{aligned} & \left[ \left( F_1(\gamma) \otimes_{\mathbb{C}} h_1(x) \right) \otimes_{\mathcal{O}(Y)} a_1, \left( F_2(\gamma) \otimes_{\mathbb{C}} h_2(x) \right) \otimes_{\mathcal{O}(Y)} a_2 \right] = \\ & \left( F_1(\gamma)F_2(\gamma) \otimes_{\mathbb{C}} h_1(x)h_2(x) \right) \otimes_{\mathcal{O}(Y)} [a_1, a_2] + \\ & \left( F_1(\gamma)(h_1 * p(a_1))[F_2](\gamma) \otimes_{\mathbb{C}} h_2(x) \right) \otimes_{\mathcal{O}(Y)} a_2 - \\ & \left( F_2(\gamma)(h_2 * p(a_2))[F_1](\gamma) \otimes_{\mathbb{C}} h_1(x) \right) \otimes_{\mathcal{O}(Y)} a_1 \end{aligned} \quad (5.1.8)$$

One sees that definition is just copied from the explicit formulas for the commutator of vector fields on the loop space (see formula 5.1.4). But for general algebroids we just take this formulas as a definition and at the moment we do not see geometric sense of the formulas above.

The main thing to check is to check that the bracket is well-defined, this can be done directly using the property: assume  $\phi(y)$  is function on  $Y$  and  $\sum_k \phi_1^k \otimes \phi_2^k$  is its pullback on  $Map(X, Y) \times X$ , assume  $\xi$  is vector field on  $Y$  and we will denote in the same way the corresponding field in  $Map(X, Y)$ ,  $f(x)$  - any function on  $X$  then the following is true:

$$\sum_k ((f * \xi)[\phi_1^k]) \otimes \phi_2^k = \sum_k (\xi[\phi_1^k]) \otimes f\phi_2^k \quad (5.1.9)$$

It may be tempting to say that it's true for any  $\xi \in \mathcal{T}_{Map(X, Y)}$  and any function  $\phi \in \mathcal{O}(Map(X, Y) \times X)$ , but it is not the case.

*Remark:* note that bracket is not well-defined, for the  $ev^*$  of the Courant algebroid, because of the absence of the Leibniz rule for the Courant algebroid. One can see the problem even in the simplest case:

$$[(1 \otimes_{\mathbb{C}} 1) \otimes_{\mathcal{O}(Y)} q_1, (1 \otimes_{\mathbb{C}} 1) \otimes_{\mathcal{O}(Y)} q_2] = (1 \otimes_{\mathbb{C}} 1) \otimes_{\mathcal{O}(Y)} [q_1, q_2]$$

This formula leads to the contradictions.

So we have proved the following theorem:



5.1.7. *Theorem:* Let  $A$  be an Lie algebroid on  $Y$ , with the anchor  $p : A \rightarrow \mathcal{T}_Y$ , then  $pr_*ev^*A$  is an algebroid on  $Map(X, Y)$  with the bracket given above and the anchor  $\bar{p} : pr_*ev^*A \rightarrow pr_*ev^*TY$  (recall that  $pr_*ev^*TY = TMap(X, Y)$ ) given by the formula:

$$\bar{p} \left( F(\gamma) \otimes_{\mathbb{C}} h(x) \right) \otimes_{\mathcal{O}(Y)} a = \left( F(\gamma) \otimes_{\mathbb{C}} h(x) \right) \otimes_{\mathcal{O}(Y)} p(a) \quad (5.1.10)$$

5.1.8. *Virasoro algebra and Courant algebroid on the  $S^1$  (after Severa).* *Remark:* The natural quotient of the standard Courant on  $S^1$  is the Virasoro Lie algebra i.e. consider  $0 \rightarrow \Omega^1 \rightarrow \mathcal{Q} \xrightarrow{\pi} \mathcal{T} \rightarrow 0$  And consider the quotient of the  $\Omega^1$  and  $\mathcal{Q}$  by the space of the exact forms: one obtains exact sequence:  $0 \rightarrow \mathbb{C} \rightarrow Vir \xrightarrow{\pi} \mathcal{T} \rightarrow 0$  where  $Vir$  is Lie algebra (not just Leibniz algebra) because  $[q_1, q_2] + [q_2, q_1]$  - is the exact form.

## 5.2. Poisson $\sigma$ -model.

5.2.1. *Lagrangian description of Poisson  $\sigma$ -model.* One starts with some manifold  $M$  (called target space (TS)) equipped with Poisson bivector  $\pi$  and the unit disc  $L = |z| < 1$  (called worldsheet (WS)). There are the fields of two kinds in the model: first maps  $X : L \rightarrow M$   $X(z, \bar{z}) = (X_1, \dots, X_n)$ , second  $\xi$  which are 1-forms on  $L$  with values in the pullback by the map  $X$  of the cotangent bundle to  $M$ , i.e.  $\xi \in \Gamma(\mathcal{T}_L^* \otimes X^*\mathcal{T}_M^*)$  or explicitly  $\xi(z, \bar{z}) = (\xi_1, \dots, \xi_n) \xi_{k,z} dz + \xi_{k,\bar{z}} d\bar{z}$ .

The action functional in the model is given by:  $S(X, \xi) = \int_L \xi_j dX^j + \pi_{mn} \xi_m \xi_n$ .

Or the same can be rephrased in coordinateless fashion as:  $S(X, \xi) = \int_L \langle \xi, d^t X \rangle + \langle \xi \otimes \xi, X^*(\pi) \rangle$ .

Where we mean by  $d^t X$  the differential of the map  $X$  considered as the section of  $\Gamma(\mathcal{T}_L^* \otimes X^*\mathcal{T}_M)$  so we have natural pairing with the values in 2-forms on  $L$  between  $d^t X$  and  $\xi$ ; in the second term we mean the pairing of  $\xi \otimes \xi$  with the pullback by the map  $X$  of the bivector  $\pi$ .

The insight of Kontsevich was that such model gives associative star-product.

5.2.2. *Hamiltonian description and loop space.* In order to obtain the Hamiltonian description of the model one should separate the time variable. Let  $(t, \phi)$  be the polar coordinates and one can take  $t$  to be the time variable. Hence the phase space (the space of fields when  $t$  is fixed) of the model consists of the maps  $X : S^1 \rightarrow M$  and  $\xi \in \Gamma(\mathcal{T}^*S^1 \otimes X^*\mathcal{T}_M)$ .

*Remark:* hence the space of fields  $X$  is just the loop space on  $M$ , it is configuration space, the whole phase space is cotangent bundle to the loop space (as it was explained in previous section  $pr_*(ev^*T^*M \otimes T^*S^1) = T^*LM$ ).

5.2.3. *Lie algebroid on the loop space and Poisson  $\sigma$ -model.* Recall the diagram:

$$\begin{array}{ccc} LM \times S^1 & \xrightarrow{ev} & M \\ \text{pr} \downarrow & & \\ LM & & \end{array} \quad (5.2.1)$$

where  $LM$  is free loop space for  $M$ . In the previous sections we have explained that there is well-defined Lie algebroid structure on the  $pr_*ev^*A$ , where  $A$  is any Lie algebroid on  $M$ . Let us take  $A = T^*M$  with the Lie algebroid structure induced by Poisson bivector  $\pi$  (see section 2.3).

*Claim:* Lie algebroid  $pr_*ev^*T^*M$  is the same as Lie algebroid of symmetries of Poisson  $\sigma$ -model ([17] section 5.2 )

Let us explain the claim. Choose the coordinates  $X^i$  on  $M$ . The 1-forms  $dX^i$  gives trivialization of the cotangent bundle. The elements of  $pr_*ev^*T^*M$  can be represented as follows:  $\epsilon = \sum_{k=1}^n F_k(\gamma) \otimes_{\mathbb{C}} h_k(\phi) \otimes_{\mathbb{C}} dX^k$ , where  $\gamma \in LM$  and  $\phi \in S^1$ . Denote by  $\epsilon_k(\phi) = F_k(\gamma)h_k(\phi)$ . So for the fixed  $\phi$  one can consider  $\epsilon_k(\phi)$  as a function on  $LM$ . For  $\phi \in S^1$  let us denote by  $X^k(\phi)$  the function on the loop space given by  $\gamma \mapsto X^k(\gamma(\phi))$  - just the evaluation of the  $k$ -th coordinate of the map  $\gamma$  at point  $\phi$ .

*Lemma:* the action of the anchor  $\bar{p}(\epsilon)$  on the function  $X^k(\phi)$  is obviously given by the formula:

$$\bar{p}(\epsilon)(X^k(\phi)) = \sum_j \pi^{k,j}(\gamma(\phi))^{k,j} \epsilon_j(\phi) \quad (5.2.2)$$

This is the same as the formula 5.5 from [17] (the anchor was denoted by  $\delta_\epsilon$ , our map  $\gamma$  was denoted by  $X$ , the point  $\phi$  is usually omitted in physical notations). So one sees that our anchor coincides with the anchor for the Lie algebroid from [17].

The bracket between 1-forms  $dX^i \in T^*M$  is obviously given by the formula  $[dX^i, dX^j] = \partial_k \pi^{i,j} dX^k$ . Hence we have the following lemma.

*Lemma:* For the elements  $\epsilon \in pr_*ev^*T^*M$  of the special kind:  $\epsilon = \sum_{k=1}^n 1 \otimes_{\mathbb{C}} h_k(\phi) \otimes_{\mathbb{C}} dX^k$  the bracket is obviously given by the formula:

$$[\epsilon, \epsilon'] = \partial_k \pi^{i,j} h_i(\phi) h'_j(\phi) dX^k \quad (5.2.3)$$

This is the same bracket which was introduced in [17] (see the first formula in section 5.2 of [17] (the fact that this is the bracket only on the elements  $\epsilon$  of the special kind was somehow not mentioned there)).

In paper [17] the action of this algebroid on the space  $T^*LM$  was modified by the 1-cocycle (see formula 5.6) at the moment we do not have an interpretation of this fact in our approach.

So we have argued that our general and quite simple construction in particular case gives the same algebroid as Lie algebroid of symmetries of Poisson  $\sigma$ -model, considered from the other motivation in [17].

**5.2.4. Conjectural generalization to WZNW-Poisson  $\sigma$ -model and Courant algebroid.** In analogy to the all above it would be tempting to propose the following: consider the Courant obtained by twisting the standard algebroid by the 3-form  $H$  and consider the Dirac structure to corresponding to WZNW-Poisson condition (see section 4.5), then it is Lie algebroid, one can apply to it our construction and obtain some Lie algebroid on the loop space, one can hope that this algebroid plays the role of symmetries for the WZNW-Poisson  $\sigma$ -model.

## Appendixes

### A.1 Some formulas.

The following formulas holds true:

$$[d, \iota_{\xi_1} \iota_{\xi_2}] = \iota_{-[\xi_1, \xi_2]} + \mathcal{L}_{\xi_1} \iota_{\xi_2} - \mathcal{L}_{\xi_2} \iota_{\xi_1}, \quad (\text{A.1})$$

$$[d, \iota_{\xi_1} \iota_{\xi_2} \iota_{\xi_3}]_+ = \mathcal{L}_{\xi_1} \iota_{\xi_2} \iota_{\xi_3} + c.p. + \iota_{\xi_1} \iota_{[\xi_2, \xi_3]} + c.p. \quad (\text{A.2})$$

Let us recall that we denoted by  $\langle, \rangle$  the contraction of polyvectors and forms. It coincides with  $\iota_{\xi} \omega$  for  $\deg \xi < \deg \omega$ , but by conventions  $\iota_{\xi} \omega = 0$ , for  $\deg \xi > \deg \omega$ . And it is not the case for  $\langle, \rangle$ . Recall that generalized Lie derivative  $\mathcal{L}_{\xi}$  action of polyvectors on forms is defined by the Cartan formula:  $\mathcal{L}_{\xi} \omega = [\iota_{\xi}, d]_{\text{graded}} \omega$ . Let us modify the definition:

$$\mathcal{L}_{\xi}^{\text{mod}} \omega = \langle \xi, d\omega \rangle + (-1)^{\deg \xi + 1} d \langle \xi, \omega \rangle \quad (\text{A.3})$$

i.e. we changed the  $\iota_{\xi}$  to contraction with  $\xi : \langle \xi, \bullet \rangle$ .

Let us denote by  $[polyvector, polyvector]$  the usual Schouten bracket for the polyvectors and  $[polyvector, form]$  - the modified Lie derivative action of polyvectors on forms.

The contraction and commutator are consistent in the sense that the following holds:

$$[\xi, \langle \nu, \omega \rangle] = \langle [\xi, \nu], \omega \rangle + \langle \nu, [\xi, \omega] \rangle \quad (\text{A.4})$$

It is the analog of the usual  $[\mathcal{L}_{\xi}, \iota_{\nu}] = \iota_{[\xi, \nu]}$ .

For example:

$$\begin{aligned} \langle \pi, \alpha \rangle, \langle \pi, \beta \rangle &= \langle [\pi, \langle \pi, \beta \rangle], \alpha \rangle - \langle \pi, \mathcal{L}_{\langle \pi, \beta \rangle} \alpha \rangle \\ [\pi, \langle \pi, \beta \rangle] &= \langle [\pi, \pi], \beta \rangle + \langle \pi, [\pi, \beta] \rangle \end{aligned} \quad (\text{A.5})$$

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